A NOTE ON THE λ-STRUCTURE ON THE BURNSIDE RING

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Abstract. Let G be a finite group and let S be a G-set. The Burnside ring of G has a natural structure of a λ-ring, $\{\lambda^n\}_{n\in\mathbb{N}}$. However, a priori $\lambda^n(S)$ can only be computed recursively, by first computing $\lambda^1(S),\dots,\lambda^{n-1}(S)$. In this paper we establish an explicit formula, expressing $\lambda^n(S)$ as a linear combination of classes of G-sets.

1. Introduction

We use $\mathcal{B}(G)$ to denote the Burnside ring of the finite group G. Recall that, as an abelian group, $\mathcal{B}(G)$ is free on $\{[S]\}_{S\in R}$, where R is a set of representatives of the isomorphism classes of transitive G-sets, and that its rank equals the number of conjugacy classes of subgroups of G. When f is a function on $\mathcal{B}(G)$, we write $f(S)$ for $f([S])$.

There is a λ-structure on $\mathcal{B}(G)$, $\{\lambda^n\}_{n\in\mathbb{N}}$, defined as the opposite structure of $\{\sigma^n\}_{n\in\mathbb{N}}$, where $\sigma^n(S)$ is the class of the nth symmetric power of S.[1] It should be considered the natural λ-structure on $\mathcal{B}(G)$, one reason for this being that there is a canonical homomorphism to the ring of rational representations of G, $\mathrm{h}\colon \mathcal{B}(G)\to \mathrm{R}_{\mathbb{Q}}(G)$, defined by $\mathrm{h}(S)=\big[\mathbb{Q}[S]\big]$, and the given λ-structure on $\mathcal{B}(G)$ makes h into a λ-homomorphism. (Note however that this λ-structure is non-special.)

The implicit nature of the definition of the λ-structure on $\mathcal{B}(G)$ makes it hard to compute with. The main result of this paper is a closed formula for $\lambda^i(S)$, where S is any G-set. To state it we first introduce some notation:

Definition 1.1. *Let $\mu\vdash i$, i.e., μ is a partition of i. We use $\ell(\mu)$ to denote the length of μ. Also, if $\mu=(\mu_1,\dots,\mu_l)$, where $\mu_1=\dots=\mu_{\alpha_1}>\mu_{\alpha_1+1}=\dots=\mu_{\alpha_1+\alpha_2}>\dots>\mu_{l-\alpha_{l'}+1}=\dots=\mu_l$ we define the tuple $\alpha(\mu):=(\alpha_1,\dots,\alpha_{l'})$, and write $\binom{l(\mu)}{\alpha(\mu)}$ for $\frac{l!}{\alpha_1!\cdots\alpha_{l'}!}$. Finally, for S a G-set of cardinality $\geq i$ we define $\mathcal{P}_\mu(S)$ to be the G-set consisting of $\ell(\mu)$-tuples of pairwise disjoint subsets of S, where the first one has cardinality μ_1, and so on.*

Using this notation we can express $\lambda^n(S)$, for any G-set S, as a linear combination of classes of G-sets:

$$\lambda^i(S)=(-1)^i\sum_{\mu\vdash i}(-1)^{\ell(\mu)}\binom{\ell(\mu)}{\alpha(\mu)}[\mathcal{P}_\mu(S)]\in\mathcal{B}(G)\tag{1.2}$$

when $i\leq|S|$, and $\lambda^i(S)=0$ when $i>|S|$. When S is transitive, so are the $\mathcal{P}_\mu(S)$, so in this case, (1.2) expresses $\lambda^i(S)$ as a linear combination of different elements from the standard basis of $\mathcal{B}(G)$.

We will prove (1.2) in the following way: First, recall that a group homomorphism $\phi\colon H\to G$ gives rise to a λ-homomorphism $\mathrm{res}^G_H\colon \mathcal{B}(G)\to\mathcal{B}(H)$ by restricting the action on a G-set S to an H-action via ϕ. Now let G be a finite group and let S be a G-set of cardinality n. By choosing an enumeration of S we get a homomorphism $G\to\Sigma_n$, the symmetric group on $\{1,\dots,n\}$. Let $\mathrm{res}^{\Sigma_n}_G$ be the corresponding restriction homomorphism (which is independent of the chosen enumeration). We have that $\mathrm{res}^{\Sigma_n}_G\big(\{1,\dots,n\}\big)=[S]$, hence $\mathrm{res}^{\Sigma_n}_G\big(\lambda^i(\{1,\dots,n\})\big)=\lambda^i(S)$. Also, writing $\mathcal{P}^{(n)}_\mu$ for $\mathcal{P}_\mu(\{1,\dots,n\})$, we see that $\mathrm{res}^{\Sigma_n}_G(\mathcal{P}^{(n)}_\mu)=[\mathcal{P}_\mu(S)]$. Hence, to prove (1.2) it suffices to prove it in the special case when $G=\Sigma_n$ and $S=\{1,\dots,n\}$,

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[1]Let $\{\sigma^n\}_{n\in\mathbb{N}}$ be a λ-structure on the ring R and define $\sigma_t(x):=\sum_{i\geq0}\sigma^i(x)t^i\in R[[t]]$. The λ-structure opposite to $\{\sigma^n\}$ is defined by $\sigma_t(x)\cdot\lambda_{-t}(x)=1\in R[[t]]$, where $\lambda_t(x):=\sum_{i\geq0}\lambda^i(x)t^i$.

i.e.,

$$\lambda^i\big(\{1,\dots,n\}\big) = (-1)^i \sum_{\mu\vdash i} (-1)^{\ell(\mu)} \binom{\ell(\mu)}{\alpha(\mu)} [\mathcal{P}_\mu^{(n)}] \in \mathcal{B}(\Sigma_n). \tag{1.3}$$

The validity of (1.3) will be established in Theorem 3.6.

We have not been able to derive (1.3) completely inside of the Burnside ring. Instead we have to use the canonical λ-homomorphism $\mathrm{h}\colon \mathcal{B}(\Sigma_n) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ to move some of the computations to the rational representation ring, whose λ-structure is much easier to work with. However, there is a problem in that h is not injective for Σ_n. In Section 2 we will therefore introduce a subring $Schur_n$ of $\mathcal{B}(\Sigma_n)$, with the property that the restriction of $\mathrm{h}\colon \mathcal{B}(\Sigma_n) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ to $Schur_n$ is injective.

In Section 3 we then establish (1.3). The technique of passing to the representation ring will be used at a crucial place, to prove Lemma 3.4.

In a forthcoming paper, [RÖ7a], we give an application of (1.3): We use it to derive a formula for the class of a certain torus in the Grothendieck ring of varieties.

An introduction to λ-rings, representation rings and the Burnside ring is given in [Knu73]. The standard reference for λ-rings is [AT69], whereas representation rings are extensively studied in [Ser77].

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2. The Schur subring of $\mathcal{B}(\Sigma_n)$

Recall that we write $\mathcal{P}_\mu^{(n)}$ for $\mathcal{P}_\mu(\{1,\dots,n\}) \in \mathcal{B}(\Sigma_n)$.

Let S be a Σ_n-set. We say that S is a *Schur set* if for every $s \in S$, $(\Sigma_n)_s$, the stabilizer subgroup of s, is a Schur subgroup, i.e., it is the stabilizer of some partition of $\{1,\dots,n\}$. Equivalently, any transitive component of S is isomorphic to $\mathcal{P}_\mu^{(n)}$ for some $\mu \vdash n$.

Definition 2.1. *$Schur_n$ is the subgroup of $\mathcal{B}(\Sigma_n)$ generated by the Schur sets.*

Equivalently, this means that $Schur_n \subset \mathcal{B}(\Sigma_n)$ is the free subgroup on $\{[\mathcal{P}_\mu]\}_{\mu\vdash n}$. The reason for us to introduce $Schur_n$ is the next theorem:

Theorem 2.2. *Let $\mathrm{h}\colon \mathcal{B}(\Sigma_n) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ be the canonical λ-ring homomorphism. The restriction of* h *to $Schur_n$ is injective.*

Even though this is a simple consequence of the injectivity of the character homomorphism from $\mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ to the ring of symmetric polynomials, we have chosen to give a more direct proof:

Proof. For every $\mu \vdash n$, let $\sigma_\mu \in \Sigma_n$ be an element in the conjugacy class determined by μ and let $C_{\sigma_\mu}\colon \mathrm{R}_{\mathbb{Q}}(\Sigma_n) \to \mathbb{Z}$ be the homomorphism defined by $V \mapsto \chi_V(\sigma_\mu)$, where χ_V is the character of V. This definition is independent of the choice of σ_μ. Together the C_{σ_μ} give a homomorphism

$$\mathrm{R}_{\mathbb{Q}}(\Sigma_n) \to \prod_{\mu\vdash n} \mathbb{Z},$$

and it suffices to show that the composition of this with the restriction of h to $Schur_n$ is injective, i.e., that

$$\begin{aligned} \varphi\colon Schur_n &\to \prod_{\mu\vdash n} \mathbb{Z} \\ [T] &\mapsto \big(|T^{\sigma_\mu}|\big)_{\mu\vdash n} \end{aligned}$$

is injective, where T^{σ_μ} is the set of points in T fixed by σ_μ. To do this, define a total ordering on the set of partitions of n by $\mu > \mu'$ if $\mu_1 = \mu'_1, \dots, \mu_{j-1} = \mu'_{j-1}$ and $\mu_j > \mu'_j$ for some j (i.e., lexicographic order). We claim that $|\mathcal{P}_\mu^{\sigma_\mu}| \neq 0$, whereas $\mathcal{P}_{\mu'}^{\sigma_\mu} = \emptyset$ if $\mu > \mu'$. (Here and in the rest of this proof we write $\mathcal{P}_\mu$ for $\mathcal{P}_\mu^{(n)}$.)

For the first assertion, choose for example

$$\sigma_\mu = (1,\dots,\mu_1)(\mu_1+1,\dots,\mu_1+\mu_2)\cdots(n-\mu_{\ell(\mu)}+1,\dots,n).$$

Then

$$\big(\{1,\dots,\mu_1\},\{\mu_1+1,\dots,\mu_1+\mu_2\},\dots,\{n-\mu_{\ell(\mu)}+1,\dots,n\}\big)\in\mathcal{P}_\mu$$

is fixed by σ_μ.

For the second assertion, suppose $\mu'<\mu$ and $t=(T_1,\dots,T_l)\in\mathcal{P}_{\mu'}$, where $l=\ell(\mu')$. Suppose moreover that t is fixed by σ_μ. If now $\mu_1>\mu_2>\dots>\mu_{\ell(\mu)}$, then, with the same σ_μ as above, we must have $T_1=\{1,\dots,\mu_1\},\dots,T_l=\{n-\mu_l+1,\dots,n\}$. (This is because $\mu_j\geq\mu'_j$ for every j and if 1 lies in T_j then so does $\sigma_\mu(1)=2$, hence also $3,\dots,\mu_1$. So T_j has cardinality at least μ_1 and the only μ'_j that can be that big is μ'_1.) But if μ and μ' differ in position j it is impossible for T_j to fulfill this since it has cardinality $\mu'_j<\mu_j$. In the general case, when we may have $\mu_j=\mu_{j+1}$, the above argument works the same only that we for example can have $T_1=\{\mu_1+1,\dots,\mu_1+\mu_2\}$ and $T_2=\{1,\dots,\mu_1\}$ if $\mu_1=\mu_2$.

We are now ready to prove that φ is injective. Let $x=\sum_{\mu\vdash n}a_\mu[\mathcal{P}_\mu]$, where $a_\mu\in\mathbb{Z}$, and suppose that $x\neq 0$. Choose the maximal μ_0 such that $a_{\mu_0}\neq 0$. Let φ_{μ_0} be the μ_0th component of φ. Then

$$\varphi_{\mu_0}(x)=\sum_{\mu\vdash n}a_\mu|\mathcal{P}_\mu^{\sigma_{\mu_0}}|=a_{\mu_0}|\mathcal{P}_{\mu_0}^{\sigma_{\mu_0}}|\neq 0;$$

hence $\varphi(x)\neq 0$. □

We conclude this section by showing that $Schur_n$ is a subring of $\mathcal{B}(\Sigma_n)$.

Proposition 2.3. *$Schur_n$ is closed under multiplication.*

Proof. We want to see what happens when we multiply $[S]$ and $[T]$, where S and T are Schur sets. Let $s\in S$ and $t\in T$. Then the stabilizers of s and t equal the stabilizers of partitions of $\{1,\dots,n\}$, which we denote $(S_1,\dots,S_k)$ and $(T_1,\dots,T_l)$, respectively. Then $\sigma\in\Sigma_n$ is in the stabilizer of (s,t) precisely when $\sigma S_i=S_i$ and $\sigma T_j=T_j$ for each i,j. Equivalently, σ must preserve $S_i\cap T_j$ for each i,j. Hence $(\Sigma_n)_{(s,t)}$ equals the stabilizer of the partition $\{S_i\cap T_j\}_{i,j}$. Consequently, it is a Schur subgroup, hence $S\times T$ is a Schur set. Therefore $Schur_n$ is closed under multiplication. □

Remark. $Schur_n$ is in general not a λ-ring since it is not closed under the λ-operations. For example, note that when S is a Σ_n-set we can represent the symmetric square of S as the set of 2-subsets of S. Now, let S be the Σ_4-set $\{1,2,3,4\}$ and consider $x:=\sigma^2\big(\sigma^2(S)\big)\in\mathcal{B}(\Sigma_4)$. An element of the underlying Σ_4-set is $\{\{1,2\},\{3,4\}\}$ and the stabilizer G of this element is generated by $\{(12),(34),(13)(24),(14)(23)\}$. The only partition that is stabilized by G is the trivial one, so since G does not equal Σ_4 it fails to be the stabilizer of a partition. Hence $x\notin Schur_4$. Since $\lambda^2\big(\sigma^2(S)\big)=\big(\sigma^2(S)\big)^2-x$ it follows that this is not contained in $Schur_4$ either. But $\sigma^2(S)$ is in $Schur_4$, which is therefore not a λ-ring. This also gives an example showing that $\mathcal{B}(\Sigma_4)$ is not special. For if it were, then $y:=\lambda^2\big(\lambda^2(S)\big)$ would be a polynomial in $\lambda^i(S)$ for $i=1,2,3,4$, which all lie in $Schur_4$, so y would also lie in $Schur_4$. But using the above one shows that $y\notin Schur_4$.

3. The λ-operations on $\mathcal{B}(\Sigma_n)$

We are now ready to start the investigation of how λ^i acts on $\{1,\dots,n\}$, the goal being to obtain a closed formula for it. We will need some more definitions. We have only defined $\mathcal{P}_\mu^{(n)}$ when μ is a partition of $i\leq n$. More generally:

Definition 3.1. *If $\alpha=(i_1,\dots,i_l)$ is any tuple of positive integers summing up to $i\leq n$ we define $\mathcal{P}_\alpha^{(n)}$ to be the Σ_n-set of l-tuples of disjoint subsets of $\{1,\dots,n\}$, the first one having i_1 elements, and so on.*

We have $[\mathcal{P}_\alpha^{(n)}]=[\mathcal{P}_\mu^{(n)}]$, where μ is the i-tuple corresponding to α. Also note that $[\mathcal{P}_\alpha^{(n)}]=[\mathcal{P}_{\alpha,n-i}^{(n)}]$, where we use $\mathcal{P}_{\alpha,n-i}^{(n)}$ to denote $\mathcal{P}_{(i_i,\dots,i_l,n-i)}^{(n)}$. Similarly, if $\beta=(j_1,\dots,j_k)$ we will write $\mathcal{P}_{\alpha,\beta}^{(n)}$ for $\mathcal{P}_{(i_1,\dots,i_l,j_1,\dots,j_k)}^{(n)}$.

Throughout this section we use the following notation:

$$\begin{aligned}s_i^{(n)}&:=\sigma^i(\{1,\dots,n\})\\ \ell_i^{(n)}&:=\lambda^i(\{1,\dots,n\})\in\mathcal{B}(\Sigma_n)\end{aligned}$$

We begin by giving a formula for $s_i^{(n)}$ which shows that it lies in $Schur_n$, and we then deduce from this that also $\ell_i^{(n)}$ is in $Schur_n$. Recall from the introduction that if $\mu = (\mu_1, , \dots, \mu_l)$, where $\mu_1 = \cdots = \mu_{\alpha_1} > \mu_{\alpha_1+1} = \cdots = \mu_{\alpha_1+\alpha_2} > \cdots > \mu_{j-\alpha_{l'}+1} = \cdots = \mu_j$, then we define $\alpha(\mu) := (\alpha_1, \dots, \alpha_{l'})$.

Proposition 3.2. *We have*

$$s_i^{(n)} = \sum_{\substack{\mu \vdash i: \\ \ell(\mu) \leq n}} [\mathcal{P}^{(n)}_{\alpha(\mu)}].$$

In particular, $s_i^{(n)}$ and $\ell_i^{(n)}$ are in $Schur_n$ for every i.

Proof. Identify $\{1, \dots, n\}$ with $\{x_1, \dots, x_n\}$. Then the symmetric ith power of $\{1, \dots, n\}$, the Σ_n-set $\{1, \dots, n\}^i/\Sigma_i$, is identified with the set of monomials

$$\{x_1^{e_1} \cdots x_n^{e_n} : e_1 + \cdots + e_n = i\} = \dot{\bigcup}_{\substack{e_1+\cdots+e_n=i \\ e_1 \geq e_2 \cdots \geq e_n \geq 0}} \Sigma_n \cdot x_1^{e_1} \cdots x_n^{e_n},$$

where the index set on the disjoint union can be identified with the set of $\mu \vdash i$ such that $\ell(\mu) \leq n$. Now let $e_1 = \cdots = e_{\alpha_1} > e_{\alpha_1+1} = \cdots = e_{\alpha_1+\alpha_2} > \cdots > e_{n-\alpha_l+1} = \cdots = e_n$. Then

$$\begin{aligned}\Sigma_n \cdot x_1^{e_1} \cdots x_n^{e_n} =& \Sigma_n \cdot (x_1 \cdots x_{\alpha_1})^{e_1} (x_{\alpha_1+1} \cdots x_{\alpha_1+\alpha_2})^{e_{\alpha_1+1}} \cdots (x_{n-\alpha_l+1} \cdots x_n)^{e_{n-\alpha_l+1}} \\ \simeq& \Sigma_n \big(\{x_1, \dots, x_{\alpha_1}\}, \{x_{\alpha_1+1}, \dots, x_{\alpha_1+\alpha_2}\}, \dots, \{x_{n-\alpha_l+1}, \dots, x_n\}\big) \\ \simeq& \mathcal{P}^{(n)}_{(\alpha_1, \dots, \alpha_l)}\end{aligned}$$

so the first part of the proposition follows.

To show that also $\ell_i^{(n)} \in Schur_n$ we use that, by definition,

$$-(-1)^i \ell_i^{(n)} = \sum_{j=0}^{i-1} (-1)^j \ell_j^{(n)} s_{i-j}^{(n)}.$$

Since we know that $Schur_n$ is a ring, and that all $s_j^{(n)}$ and $\ell_1^{(n)} = [\mathcal{P}_1^{(n)}]$ are in $Schur_n$, it follows by induction that $\ell_i^{(n)} \in Schur_n$. □

Recall the definition of the induction map, analogous to that for representation rings: If $\phi \colon H \to G$ is a homomorphism of groups and S is a H-set, then we can associate to it the G-set $G \times_H S$, i.e., the quotient of $G \times S$ by the equivalence relation $(g \cdot \phi(h), s) \sim (g, hs)$ for $(g, s) \in G \times S$ and $h \in H$, with a G-action given by $g' \cdot (g, s) := (g'g, s)$. This gives rise to the induction map $\mathrm{ind}_H^G \colon \mathcal{B}(H) \to \mathcal{B}(G)$, which is additive but not multiplicative. We will only use it in the case when H is a subgroup of G. In this case, note that if we choose a set of coset representatives of G/H, say $R = \{g_1, \dots, g_r\}$, then we can represent $G \times_H S$ as $R \times S$ with G-action given by $g \cdot (g_i, s) = (g_j, hs)$, where $gg_i = g_j h$ for $h \in H$. The map $\mathrm{h} \colon \mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$ commutes with the induction maps if H is a subgroup of G.

In the following two lemmas we show that $\ell_i^{(n)}$ and $[\mathcal{P}_\mu^{(n)}]$, where $\mu \vdash i$, are determined by $\ell_i^{(i)}$ and $[\mathcal{P}_\mu^{(i)}]$ respectively. For this we use the map

$$\mathrm{ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \circ \mathrm{res}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_i} \colon \mathcal{B}(\Sigma_i) \to \mathcal{B}(\Sigma_n)$$

which is constructed in the following way: We view Σ_i as the symmetric group on $\{1, \dots, i\}$ and embed it in Σ_n, the symmetric group on $\{1, \dots, n\}$. Moreover we view Σ_{n-i} as the symmetric group on $\{i + 1, \dots, n\}$. We then restrict from $\mathcal{B}(\Sigma_i)$ to $\mathcal{B}(\Sigma_i \times \Sigma_{n-i})$ with respect to the projection $\Sigma_i \times \Sigma_{n-i} \to \Sigma_i$ and we induce from $\mathcal{B}(\Sigma_i \times \Sigma_{n-i})$ to $\mathcal{B}(\Sigma_n)$ with respect to the inclusion $(\tau, \rho) \mapsto \tau\rho = \rho\tau \colon \Sigma_i \times \Sigma_{n-i} \to \Sigma_n$.

Lemma 3.3. *Let $\mu \vdash i$. For $n \geq i$, $\mathrm{ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} \circ \mathrm{res}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_i} (\mathcal{P}_\mu^{(i)}) = [\mathcal{P}_\mu^{(n)}] \in \mathcal{B}(\Sigma_n)$.*

Proof. Let $R = \{\sigma_1, \dots, \sigma_r\}$, where $r = \binom{n}{i}$, be a system of coset representatives for $\Sigma_n/(\Sigma_i \times \Sigma_{n-i})$. We know that $\Sigma_n \times_{\Sigma_i \times \Sigma_{n-i}} \mathcal{P}_\mu^{(i)}$ can be identified with the set of pairs (σ_j, t), where $\sigma_j \in R$ and $t = (T_1, \dots, T_l) \in \mathcal{P}_\mu^{(i)}$. From this set we define a map to $\mathcal{P}_\mu^{(n)}$ by

$$(\sigma_j, t) \mapsto \big(\sigma_j T_1, \dots, \sigma_j T_l, \sigma_j\{i + 1, \dots, n\}\big).$$

This map is surjective for given $t' = (T'_1, \ldots, T'_l, T'_{l+1}) \in \mathcal{P}^{(n)}_\mu$, there is a $\sigma \in \Sigma_n$ such that $\sigma\{1, \ldots, \mu_1\} = T'_1, \ldots, \sigma\{i - \mu_l + 1, \ldots, i\} = T'_l$. Let $\sigma_j \in R$ be such that $\sigma = \sigma_j \tau \rho$ where $(\tau, \rho) \in \Sigma_i \times \Sigma_{n-i}$. Then

$$\big(\sigma_j, \tau(\{1, \ldots, \mu_1\}, \ldots, \{i - \mu_l + 1, \ldots, i\})\big) \mapsto t'.$$

Since both sets have $n!/(\mu_1! \cdots \mu_l!(n-i)!)$ elements this is a bijection. Finally, the map is G-equivariant, hence it is an isomorphism. □

It is the following lemma that forces us to pass to the representation ring, for we have not been able to prove it directly in the Burnside ring.

Lemma 3.4. *Given i. For $n \geq i$ we have* $\operatorname{ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_{n-i}} \circ \operatorname{res}^{\Sigma_i}_{\Sigma_i \times \Sigma_{n-i}} \big(\ell^{(i)}_i\big) = \ell^{(n)}_i \in \mathcal{B}(\Sigma_n)$.

Proof. We pass to the representation ring. Here, since h is a morphism of λ-rings that commutes with the induction and restriction maps, the image of the left hand side under h is

$$\operatorname{ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_{n-i}} \circ \operatorname{res}^{\Sigma_i}_{\Sigma_i \times \Sigma_{n-i}} \circ \lambda^i\big(\mathbb{Q}\big[\{1, \ldots, i\}\big]\big) \qquad\qquad \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n),$$

and the image of the right hand side is $\lambda^i\big(\mathbb{Q}\big[\{1, \ldots, n\}\big]\big) \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$. Now $\ell^{(i)}_i \in Schur_i$ hence, by the preceding lemma, $\operatorname{ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_{n-i}} \circ \operatorname{res}^{\Sigma_i}_{\Sigma_i \times \Sigma_{n-i}} \ell^{(i)}_i \in Schur_n$. Since also $\ell^{(n)}_i \in Schur_n$ and h is injective on $Schur_n$, it suffices to prove that

$$\operatorname{ind}^{\Sigma_n}_{\Sigma_i \times \Sigma_{n-i}} \circ \operatorname{res}^{\Sigma_i}_{\Sigma_i \times \Sigma_{n-i}} \Big(\lambda^i\big(\mathbb{Q}\big[\{1, \ldots, i\}\big]\big)\Big) = \lambda^i\big(\mathbb{Q}\big[\{1, \ldots, n\}\big]\big) \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n),$$

i.e., we have to find a Σ_n-equivariant isomorphism of $\mathbb{Q}$-vector spaces

$$\varphi\colon \mathbb{Q}[\Sigma_n] \otimes_{\mathbb{Q}[\Sigma_i \times \Sigma_{n-i}]} \bigwedge\nolimits^i \mathbb{Q}\big[\{1, \ldots, i\}\big] \to \bigwedge\nolimits^i \mathbb{Q}\big[\{1, \ldots, n\}\big].$$

This is straightforward. (It is done explicitly in [RÖ7b], Proposition 2.4.2.) □

We are now ready to prove the main theorem of this section, the formula for $\ell^{(n)}_i$. For this we first introduce a concept of degree on basis elements of $Schur_n$. Fix an n and the basis $\{[\mathcal{P}^{(n)}_\mu]\}_{\mu \vdash n}$ of $Schur_n$. For $j = 1, \ldots, k$ where $2k < n$, we say that an element $[\mathcal{P}^{(n)}_\mu]$ in the basis is of degree j if it is equal to $[\mathcal{P}^{(n)}_\nu]$ for some $\nu \vdash j$. Equivalently, this means that $n - j$ is an entry of μ. Let the degree of $[\mathcal{P}^{(n)}_n] = 1$ be zero and let the degree of the remaining elements of the basis be $k + 1$. Because $n - j > k$ the degree is well-defined.

Lemma 3.5. *Let $[\mathcal{P}^{(n)}_\alpha]$ and $[\mathcal{P}^{(n)}_\beta]$ be of degree m and m' respectively, where $m + m' \leq n/2$. Then*

$$[\mathcal{P}^{(n)}_\alpha] \cdot [\mathcal{P}^{(n)}_\beta] = [\mathcal{P}^{(n)}_{\alpha,\beta}] + \text{ terms of degree} < m + m'.$$

Proof. This is a refinement of Proposition 2.3. Let $s = (S_1, \ldots, S_l) \in \mathcal{P}^{(n)}_\alpha$ and $t = (T_1, \ldots, T_{l'}) \in \mathcal{P}^{(n)}_\beta$, where S_l and $T_{l'}$ have $n - m$ and $n - m'$ elements respectively. Then the stabilizer of $(s, t) \in \mathcal{P}^{(n)}_\alpha \times \mathcal{P}^{(n)}_\beta$ equals the stabilizer of $(S_i \cap T_j)_{i,j}$. Let $m_{ij} = |S_i \cap T_j|$ and let γ be the tuple consisting of the m_{ij}. Then the transitive component of (s, t) is $\mathcal{P}^{(n)}_\gamma$. Since $m_{ll'} \geq n - m - m' \geq n/2$ it follows that the degree of $[\mathcal{P}^{(n)}_\gamma]$ is $n - m_{ll'}$, and this is $\leq m + m'$ with equality if and only if $[\mathcal{P}^{(n)}_\gamma] = [\mathcal{P}^{(n)}_{\alpha,\beta}]$. □

Theorem 3.6. *Let i be a positive integer. Then for any $n \geq i$,*

$$\lambda^i\big(\{1, \ldots, n\}\big) = (-1)^i \sum_{\mu \vdash i} (-1)^{\ell(\mu)} \tbinom{\ell(\mu)}{\alpha(\mu)} \big[\mathcal{P}^{(n)}_\mu\big] \in \mathcal{B}(\Sigma_n).$$

Proof. For $i = 1$ the formula becomes $\ell^{(n)}_1 = \big[\mathcal{P}^{(n)}_1\big] = [\{1, \ldots, n\}]$, which is true for every n.

Given i, suppose the formula is true for every pair (i', n) where $i' < i$ and n is an arbitrary integer greater than or equal to i'. We want to show that it holds for (i, n) where n is an arbitrary integer greater than or equal to i.

Since $\ell_i^{(i)} \in Schur_i$, we have $\ell_i^{(i)} = \sum_{\mu \vdash i} a_\mu \big[\mathcal{P}_\mu^{(i)}\big]$, where the a_μ are uniquely determined. Since the induction and restriction maps are additive it follows from lemma 3.3 and lemma 3.4 that

$$\ell_i^{(n)} = \sum_{\mu \vdash i} a_\mu \big[\mathcal{P}_{\mu,n-i}^{(n)}\big] \in Schur_n \tag{3.7}$$

for every $n \geq i$. It remains to show that $a_\mu = (-1)^i (-1)^{\ell(\mu)} \binom{\ell(\mu)}{\alpha(\mu)}$ for every $\mu \vdash i$. For this, fix an n greater than $2i$ and the basis $\{[\mathcal{P}_\mu]\}_{\mu \vdash n}$ of $Schur_n$. We now use the notion of degree introduced before this theorem. By (3.7), $\ell_i^{(n)}$ is a linear combination of elements of degree i. On the other hand, by the definition of $\ell_i^{(n)}$ we have

$$-(-1)^i \ell_i^{(n)} = \sum_{j=0}^{i-1} (-1)^j \ell_j^{(n)} s_{i-j}^{(n)}. \tag{3.8}$$

By induction and the formula for $s_j^{(n)}$ from Proposition 3.2, the right hand side of (3.8) equals

$$\sum_{\mu \vdash i} [\mathcal{P}_{\alpha(\mu)}^{(n)}] + \sum_{j=1}^{i-1} (-1)^j \left((-1)^j \sum_{\mu \vdash j} (-1)^{\ell(\mu)} \binom{\ell(\mu)}{\alpha(\mu)} \big[\mathcal{P}_\mu^{(n)}\big] \right) \cdot \left(\sum_{\mu \vdash i-j} [\mathcal{P}_{\alpha(\mu)}^{(n)}] \right) \tag{3.9}$$

Since we already know that $\ell_i^{(n)}$ is zero in every degree different from i it remains to compute the degree i part of (3.9). In this expression, for every j such that $0 < j < i$ we have a product of two sums, one consisting of elements of degree j and the other one consisting of elements of degree less than or equal to $i-j$, for if $\mu \vdash i-j$ then $[\mathcal{P}_{\alpha(\mu)}]$ has degree $\leq i-j$ with equality if and only if $\mu = (1, 1, \ldots, 1)$, in which case $\alpha(\mu) = (i-j)$. If $[\mathcal{P}_\mu^{(n)}]$ has degree j and $[\mathcal{P}_{\alpha(\mu')}^{(n)}]$ has degree $m \leq i-j$ then, by Lemma 3.5,

$$[\mathcal{P}_\mu^{(n)}] \cdot [\mathcal{P}_{\alpha(\mu')}^{(n)}] = [\mathcal{P}_{\mu,\alpha(\mu')}^{(n)}] + \text{ terms of degree} < j+m.$$

Hence only the degree $i-j$ part of $\sum_{\mu \vdash i-j} [\mathcal{P}_{\alpha(\mu)}^{(n)}]$, i.e., $[\mathcal{P}_{i-j}^{(n)}]$, contributes to the degree i part of (3.9), which therefore equals

$$[\mathcal{P}_i^{(n)}] + \sum_{j=1}^{i-1} \sum_{\mu \vdash j} (-1)^{\ell(\mu)} \binom{\ell(\mu)}{\alpha(\mu)} \big[\mathcal{P}_{\mu,i-j}^{(n)}\big]. \tag{3.10}$$

We write this as a linear combination of elements in $\{[\mathcal{P}_\nu^{(n)}]\}_{\nu \vdash i}$. Fix $\nu \vdash i$ with $\ell := \ell(\nu)$ and $\alpha := \alpha(\nu) = (\alpha_1, \ldots, \alpha_t)$. If $\ell = 1$ then $[\mathcal{P}_\nu^{(n)}] = [\mathcal{P}_i^{(n)}]$, so $[\mathcal{P}_\nu^{(n)}]$ occurs once in (3.10). If $\ell > 1$ then $[\mathcal{P}_\nu^{(n)}]$ occurs first when $i-j$ equals $\nu_1 = \cdots = \nu_{\alpha_1}$; the length of μ is then $\ell - 1$ and $\alpha(\mu) = (\alpha_1 - 1, \alpha_1, \ldots, \alpha_t)$, so the coefficient in front of $[\mathcal{P}_{\mu,i-j}^{(n)}]$ is $-(-1)^\ell \alpha_1 \cdot \frac{(\ell-1)!}{\alpha!}$. Also, $[\mathcal{P}_\nu^{(n)}]$ occurs in (3.10) when $i-j$ equals $\nu_{\alpha_1+1} = \cdots = \nu_{\alpha_1+\alpha_2}$, with coefficient $-(-1)^\ell \alpha_2 \cdot \frac{(\ell-1)!}{\alpha!}$, and so on. Summing up, the coefficient in front of $[\mathcal{P}_\nu^{(n)}]$ is $-(-1)^\ell (\alpha_1 + \cdots + \alpha_t) \frac{(\ell-1)!}{\alpha!} = -(-1)^\ell \binom{\ell}{\alpha}$; hence (3.10) equals

$$-\sum_{\nu \vdash i} (-1)^{\ell(\nu)} \binom{\ell(\nu)}{\alpha(\nu)} \big[\mathcal{P}_\nu^{(n)}\big].$$

Therefore, by (3.8), $\ell_i^{(n)}$ has the desired form and by induction we are through. □

Remark. This proof starts with noting that, as a consequence of the preceding lemmas, given i it suffices to compute $\ell_i^{(n)}$ for some n in order to get the formula for every n. We then compute $\ell_i^{(n)}$ for n sufficiently large. Instead we could have computed $\ell_i^{(i)}$ by using that $\mathrm{h}(\ell_i^{(i)}) = [\mathrm{sgn}] \in \mathrm{R}_{\mathbb{Q}}(\Sigma_i)$, where sgn is the signature representation. The needed expression of [sgn] as a linear combination of permutation representations is a classical formula in the theory of representations of Σ_i. We chose to give the above proof since it is purely combinatorial in nature.

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